2024-2025
Fall Semester

Course of Power System Analysis

Elements of transmission lines for the transportation of electricity

Prof. Mario Paolone
Distributed Electrical Systems Laboratory
École Polytechnique Fédérale de Lausanne (Switzerland)

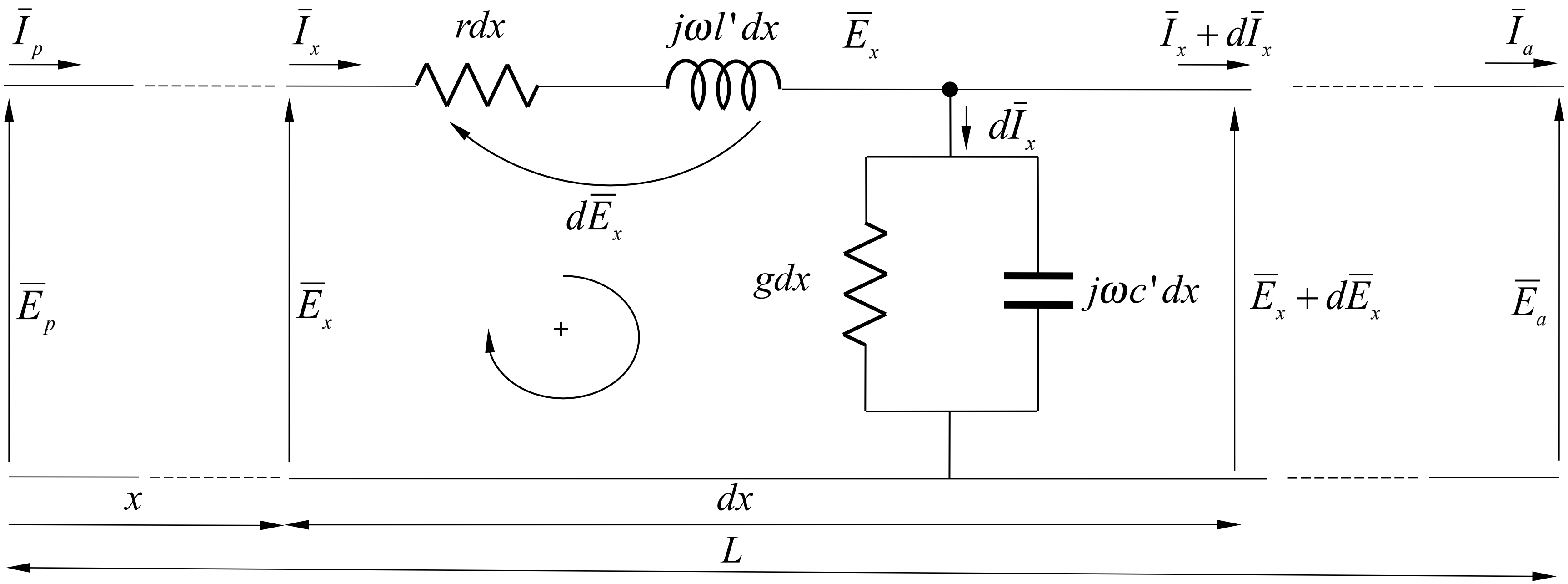
Outline

The transmission line equations

As seen in the lecture 3.1, the **distributed-parameter model** of an **electrical line** is represented by the figure here below.

Furthermore, we assume that:

1. three-phase line in a **sinusoidal steady state** conditions;
2. **three-phase system symmetrical with the current and balanced with the voltages**;
3. line characterized by a **symmetrical structure of conductors**.



Model of an electrical line for the representation with distributed parameters.

The transmission line equations

- r is the **line resistance in per unit of length**: it represents the longitudinal Joule-effect dissipative phenomena;
- l' is the **effective inductance in per unit of length**: it represents the **magnetic coupling among the different conductors** (it is an equivalent parameter for a n-conductor line);
- c' is the **effective capacity in per unit of length**: it represents the **electrical coupling among the different conductors and the ground plane** (it is an equivalent parameter for a n-conductor line);
- g is the **conductance of the line in per unit of length**: it models the transverse power losses of the line due to (i) the **corona effect** and (ii) **insulation losses** (insulators for overhead lines or insulation materials for coaxial cables).

The transmission line equations

Thus, we can define two important line parameters:

- The **longitudinal impedance in per unit of length**:

$$\bar{z} = r + j\omega\, l' \; \left[\Omega m^{-1}\right]$$

- The **transverse admittance in per unit of length**:

$$\bar{y} = g + j\omega c' \; \left[S m^{-1}\right]$$

Thanks to the representation of sinusoidal quantities in the complex domain we have:

$$e(x,t) = \overline{E}_x$$

and

$$i(x,t) = \bar{I}_x$$

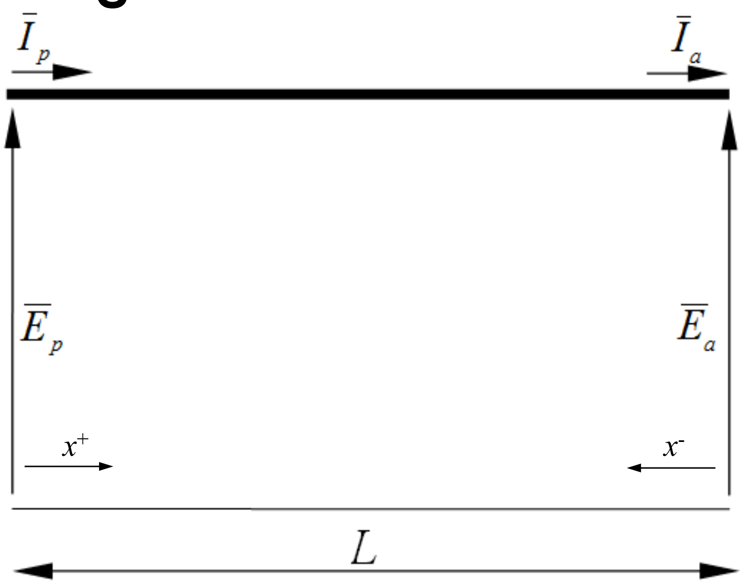
The transmission line equations

Let us recall the **transmission lines** or **telegraphers' equations** derived in the lecture 3.1.

$$\frac{d\overline{E}_x}{dx} + \overline{z}\overline{I}_x = 0$$

$$\frac{d\overline{I}_x}{dx} + \overline{y}\overline{E}_x = 0$$

In the figure below, the **boundary conditions for the current and the voltage** are given along with the **two possible orientations of the** x-axis. L is the **total line length**.



The transmission line equations

The integration of these last two equations can be obtained by using the Laplace transform (henceforth denoted as ℓ-transform):

$$\ell\left[\frac{df(x)}{dx}\right] = p\cdot\ell\left[f(x)\right] - f(0)$$

In our case we have

$$\bar{E}_x\Big|_{x=0} = \bar{E}_p \qquad\qquad \bar{I}_x\Big|_{x=0} = \bar{I}_p$$

(see figure on slide 6 the orientation x^+) and the ℓ-transform of the transmission line equations is:

$$p\cdot\ell\left[\bar{E}_x\right] - \bar{E}_p = -\bar{z}\,\ell\left[\bar{I}_x\right]$$

$$p\cdot\ell\left[\bar{I}_x\right] - \bar{I}_p = -\bar{y}\,\ell\left[\bar{E}_x\right]$$

The transmission line equations

Using the last equation, we get:

$$\ell\left[\bar{I}_x\right] = \frac{\bar{I}_p}{p} - \frac{\bar{y}}{p}\ell\left[\bar{E}_x\right]$$

That, replaced into $p \cdot \ell\left[\bar{E}_x\right] - \bar{E}_p = -\bar{z}\ell\left[\bar{I}_x\right]$ allows to obtain:

$$p \cdot \ell\left[\bar{E}_x\right] + \bar{z}\left[\frac{\bar{I}_p}{p} - \frac{\bar{y}}{p}\ell\left[\bar{E}_x\right]\right] = \bar{E}_p$$

or:

$$p^2 \cdot \ell\left[\bar{E}_x\right] + \bar{z}\bar{I}_p - \bar{z}\bar{y}\ell\left[\bar{E}_x\right] = p \cdot \bar{E}_p$$

From the last, we can derive the expression for $\ell\left[\bar{E}_x\right]$:

$$\ell\left[\bar{E}_x\right] = \frac{p \cdot \bar{E}_p - \bar{z}\bar{I}_p}{p^2 - \bar{z}\bar{y}}$$

The transmission line equations

The solution for the current is trivially obtained:

$$\ell\left[\overline{I}_x\right]=\frac{\overline{I}_p}{p}-\overline{y}\,\frac{p\cdot\overline{E}_p-\overline{z}\overline{I}_p}{p^2-\overline{z}\overline{y}}=\frac{p\cdot\overline{I}_p-\overline{y}\overline{E}_p}{p^2-\overline{z}\overline{y}}$$

Based on the definition of the line per-unit-length longitudinal impedance and transverse admittance on slide 5, we can define the following additional parameters:

- The **propagation constant**:

$$\overline{\gamma}=\sqrt{\overline{z}\,\overline{y}}=\alpha+j\beta=\sqrt{(r+j\omega l)(g+j\omega c)}\;\left[m^{-1}\right]$$

- The **characteristic impedance**:

$$\overline{Z}_0=\sqrt{\frac{\overline{z}}{\overline{y}}}=\sqrt{\frac{r+j\omega l'}{g+j\omega c'}}\;[\Omega]$$

The transmission line equations

On the basis of the previous parameters, we can rewrite the ℓ-tranform of voltage as follows:

$$\ell\left[\overline{E}_x\right]=\frac{p\cdot\overline{E}_p-\overline{\gamma}\overline{Z}_0\overline{I}_p}{p^2-\overline{\gamma}^2}$$

and the ℓ-tranform of current as:

$$\ell\left[\overline{I}_x\right]=\frac{p\cdot\overline{I}_p-\dfrac{\overline{\gamma}}{\overline{Z}_0}\overline{E}_p}{p^2-\overline{\gamma}^2}$$

The inverse ℓ-tranforms of these two last equations can be obtained by noting that:

$$\ell^{-1}\left[\frac{p}{p^2-\overline{\gamma}^2}\right]=\cosh\overline{\gamma}x \qquad \ell^{-1}\left[\frac{\overline{\gamma}}{p^2-\overline{\gamma}^2}\right]=\sinh\overline{\gamma}x$$

The expression of the current and voltage as a function of the x coordinate of the line can be finally obtained specifying the line initial conditions for $x=0$:

$$\overline{E}_x\Big|_{x=0}=\overline{E}_p \qquad \overline{I}_x\Big|_{x=0}=\overline{I}_p$$

The transmission line equations

$$\overline{E}_x = \overline{E}_p \cosh \overline{\gamma} x - \overline{Z}_0 \overline{I}_p \sinh \overline{\gamma} x$$

$$\overline{I}_x = \overline{I}_p \cosh \overline{\gamma} x - \frac{\overline{E}_p}{\overline{Z}_0} \sinh \overline{\gamma} x$$

In the case other boundary conditions are used, it is possible to use the general integral derived from the obtained solution:

$$\overline{E}_x = \left(\overline{C}_1 \cosh \overline{\gamma} x + \overline{C}_2 \sinh \overline{\gamma} x \right)$$

$$\overline{I}_x = \frac{1}{\overline{Z}_0} \left(-\overline{C}_1 \sinh \overline{\gamma} x - \overline{C}_2 \cosh \overline{\gamma} x \right)$$

or, is an equivalent form, by replacing functions $\cosh x$ and $\sinh x$ with the respective definitions:

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

The transmission line equations

$$\overline{E}_x = \overline{K}_1 e^{\overline{\gamma}\cdot x} + \overline{K}_2 e^{-\overline{\gamma}\cdot x}$$

$$\overline{I}_x = \frac{1}{\overline{Z}_0}\left(-\overline{K}_1 e^{\overline{\gamma}\cdot x} + \overline{K}_2 e^{-\overline{\gamma}\cdot x}\right)$$

The constants $\overline{C}_1, \overline{C}_2$ and $\bar{K}_1, \bar{K}_2$, are determined based on the boundary conditions. For example, if we consider the orientation x^+, the following conditions at the line start and the solution of the transmission line equations with complex exponentials, we have:

$$\overline{E}_x\Big|_{x=0} = \overline{E}_p \qquad\qquad \overline{I}_x\Big|_{x=0} = \overline{I}_p$$

and we obtain:

$$\bar{E}_p = \bar{K}_1 + \bar{K}_2$$

$$\bar{I}_p = \frac{1}{\bar{Z}_0}(-\bar{K}_1 + \bar{K}_2)$$

$$\bar{K}_1 = \frac{1}{2}\left(\bar{E}_p - \bar{Z}_0\bar{I}_p\right)$$

$$\bar{K}_2 = \frac{1}{2}\left(\bar{E}_p + \bar{Z}_0\bar{I}_p\right)$$

The transmission line equations

And the general solution for this orientation and boundary conds is:

$$\bar{E}_x = \frac{1}{2}\left(\bar{E}_p - \bar{Z}_0 \bar{I}_p\right) e^{\bar{\gamma} x} + \frac{1}{2}\left(\bar{E}_p + \bar{Z}_0 \bar{I}_p\right) e^{-\bar{\gamma} x}$$

$$\bar{I}_x = \frac{1}{2}\left(\bar{I}_p - \frac{\bar{E}_p}{\bar{Z}_0}\right) e^{\bar{\gamma} x} + \frac{1}{2}\left(\bar{I}_p + \frac{\bar{E}_p}{\bar{Z}_0}\right) e^{-\bar{\gamma} x}$$

or, by using the solutions of the transmission line equations expressed with $\cosh x$ and $\sinh x$, we have:

$$\bar{E}_x = \left(\bar{C}_1 \cosh\bar{\gamma} x + \bar{C}_2 \sinh\bar{\gamma} x\right) \qquad \bar{E}_x\Big|_{x=0} = \bar{E}_p \qquad \bar{E}_p = \bar{C}_1$$

$$\bar{I}_x = \frac{1}{\bar{Z}_0}\left(-\bar{C}_1 \sinh\bar{\gamma} x - \bar{C}_2 \cosh\bar{\gamma} x\right) \qquad \bar{I}_x\Big|_{x=0} = \bar{I}_p \qquad \bar{I}_p = \frac{1}{\bar{Z}_0}\left(-\bar{C}_2\right)$$

and the solution is:

$$\bar{E}_x = \bar{E}_p \cosh\bar{\gamma} x - \bar{Z}_0 \bar{I}_p \sinh\bar{\gamma} x$$

$$\bar{I}_x = \bar{I}_p \cosh\bar{\gamma} x - \frac{\bar{E}_p}{\bar{Z}_0} \sinh\bar{\gamma} x$$

The transmission line equations

If we **choose a reference system where the x coordinate is positive when increasing from the right to the left end of the line** (see the reference x^- on slide 6), the **solutions of the transmission line equations are the same with a change of variable $-x$ in the place of** x. Therefore, we have:

$$\bar{E}_x = \bar{C}_1 \cosh \bar{\gamma} x + \bar{C}_2 \sinh \bar{\gamma} x \xrightarrow{x=-x} \bar{E}_x = \bar{C}_1 \cosh \bar{\gamma} x - \bar{C}_2 \sinh \bar{\gamma} x$$

$$\bar{I}_x = \frac{1}{\bar{Z}_0}(-\bar{C}_1 \sinh \bar{\gamma} x - \bar{C}_2 \cosh \bar{\gamma} x) \xrightarrow{x=-x} \bar{I}_x = \frac{1}{\bar{Z}_0}(\bar{C}_1 \sinh \bar{\gamma} x - \bar{C}_2 \cosh \bar{\gamma} x)$$

and

$$\bar{E}_x = \bar{K}_1 e^{\bar{\gamma} x} + \bar{K}_2 e^{-\bar{\gamma} x} \xrightarrow{x=-x} \bar{E}_x = \bar{K}_1 e^{-\bar{\gamma} x} + \bar{K}_2 e^{\bar{\gamma} x}$$

$$\bar{I}_x = \frac{1}{\bar{Z}_0}(-\bar{K}_1 e^{\bar{\gamma} x} + \bar{K}_2 e^{-\bar{\gamma} x}) \xrightarrow{x=-x} \bar{I}_x = \frac{1}{\bar{Z}_0}(-\bar{K}_1 e^{-\bar{\gamma} x} + \bar{K}_2 e^{\bar{\gamma} x})$$

In this case, the boundary conditions for $x = 0$ are:

$$\bar{E}_x \Big|_{x=0} = \bar{E}_a \qquad \bar{I}_x \Big|_{x=0} = \bar{I}_a$$

The transmission line equations

The solutions expressed in terms of complex exponentials are:

$$\overline{E}_x = \frac{1}{2}\left(\overline{E}_a + \overline{Z}_0 \overline{I}_a\right) e^{\overline{\gamma} x} + \frac{1}{2}\left(\overline{E}_a - \overline{Z}_0 \overline{I}_a\right) e^{-\overline{\gamma} x}$$

$$\overline{I}_x = \frac{1}{2}\left(\overline{I}_a + \frac{\overline{E}_a}{\overline{Z}_0}\right) e^{\overline{\gamma} x} + \frac{1}{2}\left(\overline{I}_a - \frac{\overline{E}_a}{\overline{Z}_0}\right) e^{-\overline{\gamma} x}$$

or by using the hyperbolic functions:

$$\overline{E}_x = \overline{E}_a \cosh \overline{\gamma} x + \overline{Z}_0 \overline{I}_a \sinh \overline{\gamma} x$$

$$\overline{I}_x = \overline{I}_a \cosh \overline{\gamma} x + \frac{\overline{E}_a}{\overline{Z}_0} \sinh \overline{\gamma} x$$

Taking into account the first two equations expressing the solution of the transmission line equations with exponential functions, we can see that both voltage and current distributions are **composed by two terms**. The first one, proportional to $e^{\overline{\gamma} x}$ is called **direct wave**: it is increasing towards the positive x-coordinate. The second term, proportional to $e^{-\overline{\gamma} x}$ is called **inverse wave**: it is decreasing towards the positive x-coordinate.

The transmission line equations

Let us consider the solution of the transmission line equations as:

$$\bar{E}_x = \frac{1}{2}\left(\bar{E}_a + \bar{Z}_0 \bar{I}_a\right) e^{\bar{\gamma} x} + \frac{1}{2}\left(\bar{E}_a - \bar{Z}_0 \bar{I}_a\right) e^{-\bar{\gamma} x}$$

$$\bar{I}_x = \frac{1}{2}\left(\bar{I}_a + \frac{\bar{E}_a}{\bar{Z}_0}\right) e^{\bar{\gamma} x} + \frac{1}{2}\left(\bar{I}_a - \frac{\bar{E}_a}{\bar{Z}_0}\right) e^{-\bar{\gamma} x}$$

again, we can see that voltage and current distributions are composed by two terms: the first one, proportional to $e^{\bar{\gamma}x}$, is called **direct (stationary) wave**. The second term, proportional to $e^{-\bar{\gamma}x}$, is called **inverse (stationary) wave**.

$$\bar{E}_x = \boxed{\frac{1}{2}\left(\bar{E}_a + \bar{Z}_0 \bar{I}_a\right) e^{\bar{\gamma} x}} + \boxed{\frac{1}{2}\left(\bar{E}_a - \bar{Z}_0 \bar{I}_a\right) e^{-\bar{\gamma} x}}$$

Terms proportional to $e^{\bar{\gamma} x}$ → **direct wave**

Terms proportional to $e^{-\bar{\gamma} x}$ → **inverse wave**

$$\bar{I}_x = \boxed{\frac{1}{2}\left(\bar{I}_a + \frac{\bar{E}_a}{\bar{Z}_0}\right) e^{\bar{\gamma} x}} + \boxed{\frac{1}{2}\left(\bar{I}_a - \frac{\bar{E}_a}{\bar{Z}_0}\right) e^{-\bar{\gamma} x}}$$

The transmission line equations

By referring to the previous solutions, we define:

$$\overline{E}_d = \frac{\overline{E}_a + \overline{Z}_0 \overline{I}_a}{2}$$

$$\overline{E}_r = \frac{\overline{E}_a - \overline{Z}_0 \overline{I}_a}{2}$$

Therefore, the voltage in a generic point of the line in the x-coordinate can be calculated as:

$$\overline{E}_x = \overline{E}_d \; e^{\overline{\gamma} x} + \overline{E}_r \; e^{-\overline{\gamma} x}$$

In other words, **it is given by the superposition of the direct and the inverse stationary waves associated to the chosen boundary conditions.**

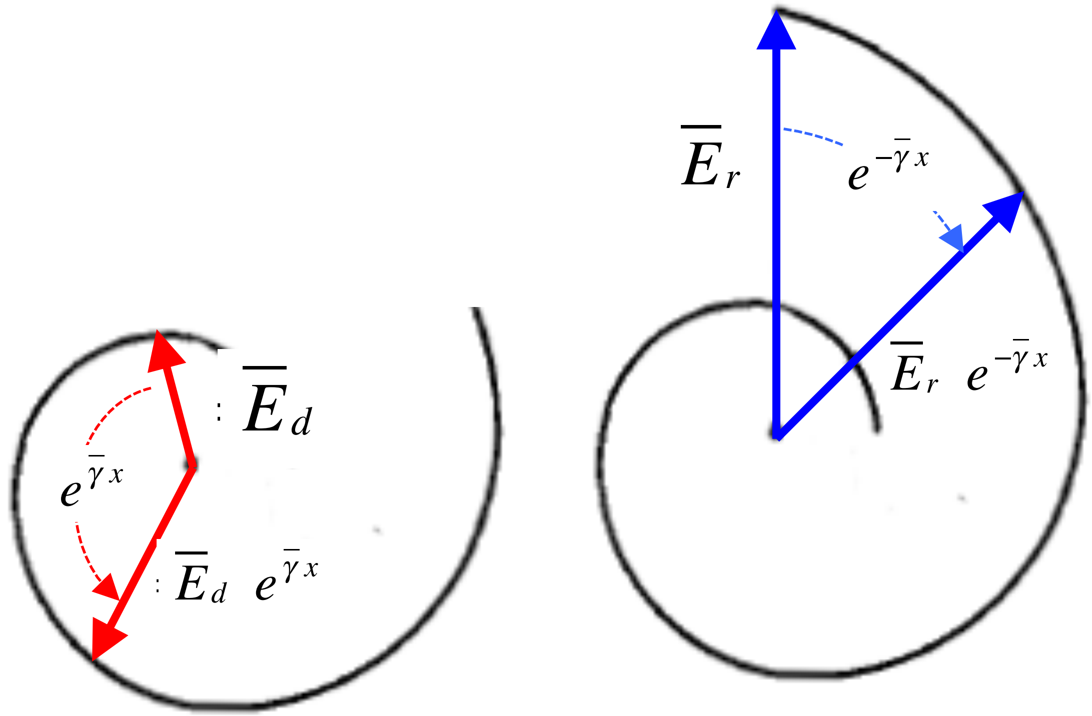
The same considerations are also valid for the current distribution:

$$\overline{I}_x = \frac{\overline{E}_d}{\overline{Z}_0} e^{\overline{\gamma} x} - \frac{\overline{E}_r}{\overline{Z}_0} e^{-\overline{\gamma} x}$$

The transmission line equations

$$\overline{E}_x = \boxed{\overline{E}_d \ e^{\overline{\gamma} x}} + \boxed{\overline{E}_r \ e^{-\overline{\gamma} x}}$$

Graphical interpretation of the superposition given by the direct and inverse stationary waves: it identifies the operating conditions of the line expressed by the voltage and current phasors profiles along the line (i.e., as function of x).



It should be noted that the **term of the inverse wave may be zero** in the case of an **infinitely long line**. This result is physically correct because, **for** $x = \infty$, **we have null electric and magnetic fields**, therefore $\bar{E}_r = 0$, $\bar{I}_r = -\frac{\bar{E}_r}{\bar{Z}_0} = 0$.

We will see later that **an infinitely long line may be realized in practice for specific boundary conditions**.

The transmission line equations

Visual interpretation of stationary waves

The transmission line equations

Let us consider an **infinitely long line** (we will see later that **an infinitely long line may be realized in practice for specific boundary conditions**).

$x \to \infty$

$\cosh \bar{\gamma} x \to \sinh \bar{\gamma} x$ and $\bar{E}_x \to 0$ (null field for $x \to \infty$)

$$\bar{E}_x = \bar{E}_p \cosh \bar{\gamma} x - \bar{Z}_0 \bar{I}_p \sinh \bar{\gamma} x \quad \to \quad \bar{E}_p - \bar{Z}_0 \bar{I}_p = 0$$

Therefore, by using the solution of the transmission line equations with complex exponentials and the orientation x^+, we have:

$$\bar{E}_x = \cancel{\frac{1}{2}\left(\bar{E}_p - \bar{Z}_0 \bar{I}_p\right) e^{\bar{\gamma} x}} + \frac{1}{2}\left(\bar{E}_p + \bar{Z}_0 \bar{I}_p\right) e^{-\bar{\gamma} x}$$

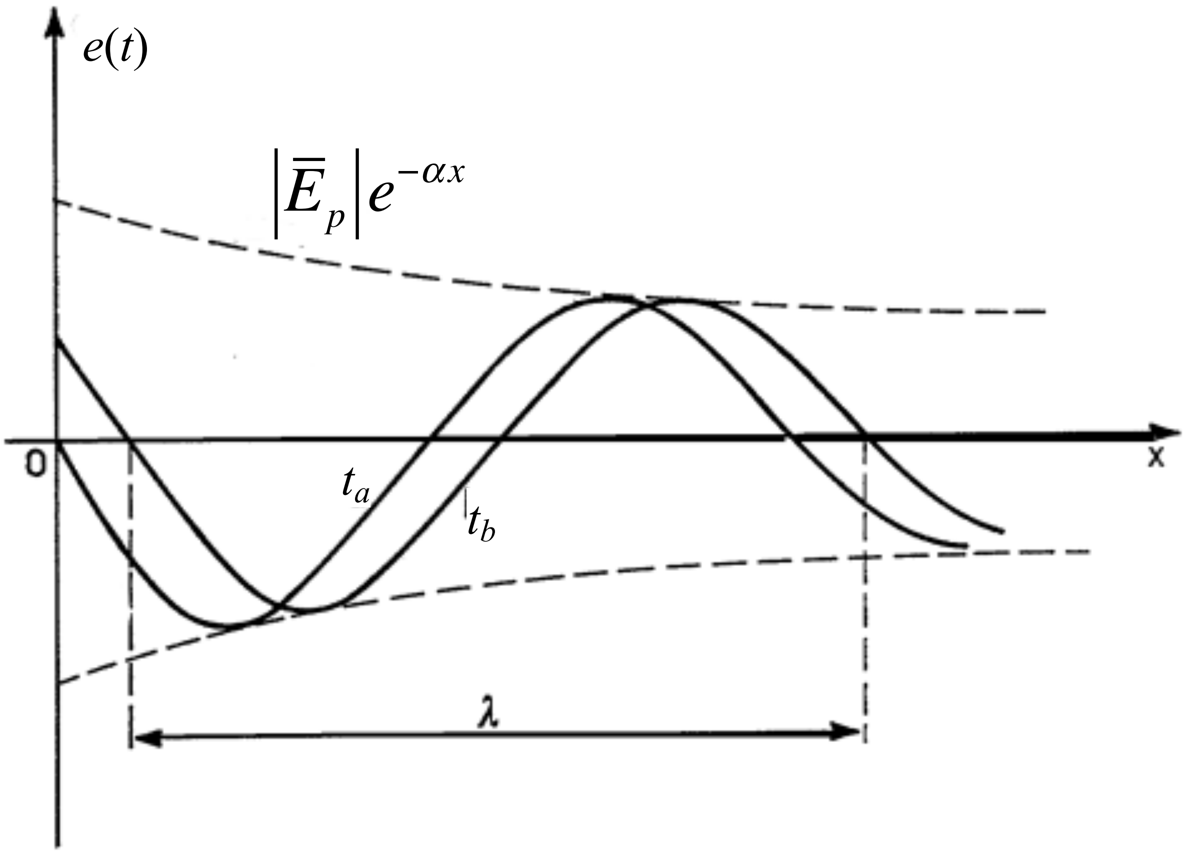
$$\bar{I}_x = \cancel{\frac{1}{2}\left(\bar{I}_p - \frac{\bar{E}_p}{\bar{Z}_0}\right) e^{\bar{\gamma} x}} + \frac{1}{2}\left(\bar{I}_p + \frac{\bar{E}_p}{\bar{Z}_0}\right) e^{-\bar{\gamma} x}$$

$$\frac{1}{2}\left(\bar{E}_p + \bar{Z}_0 \bar{I}_p\right) = \bar{E}_p \quad \to \quad \bar{E}_x = \bar{E}_p e^{-\bar{\gamma} x} = \bar{E}_p e^{-\alpha x} e^{-j\beta x}$$

$$\frac{1}{2}\left(\bar{I}_p + \frac{\bar{E}_p}{\bar{Z}_0}\right) = \bar{I}_p \quad \to \quad \bar{I}_x = \bar{I}_p e^{-\bar{\gamma} x} = \bar{I}_p e^{-\alpha x} e^{-j\beta x}$$

The transmission line equations

Therefore, if we consider an **infinitely long line**, the voltage and current phasors will progressively attenuate ($e^{-\alpha x}$) and rotate ($e^{-j\beta x}$) along the x coordinate of the line.



The graph shows the instantaneous voltage values along the line for two time instants t_a and t_a.

Obviously, the wavelength λ will be difficult to observe in practive since, as see, for a transmission line operating at 50 Hz it is in the order of thousands of km.

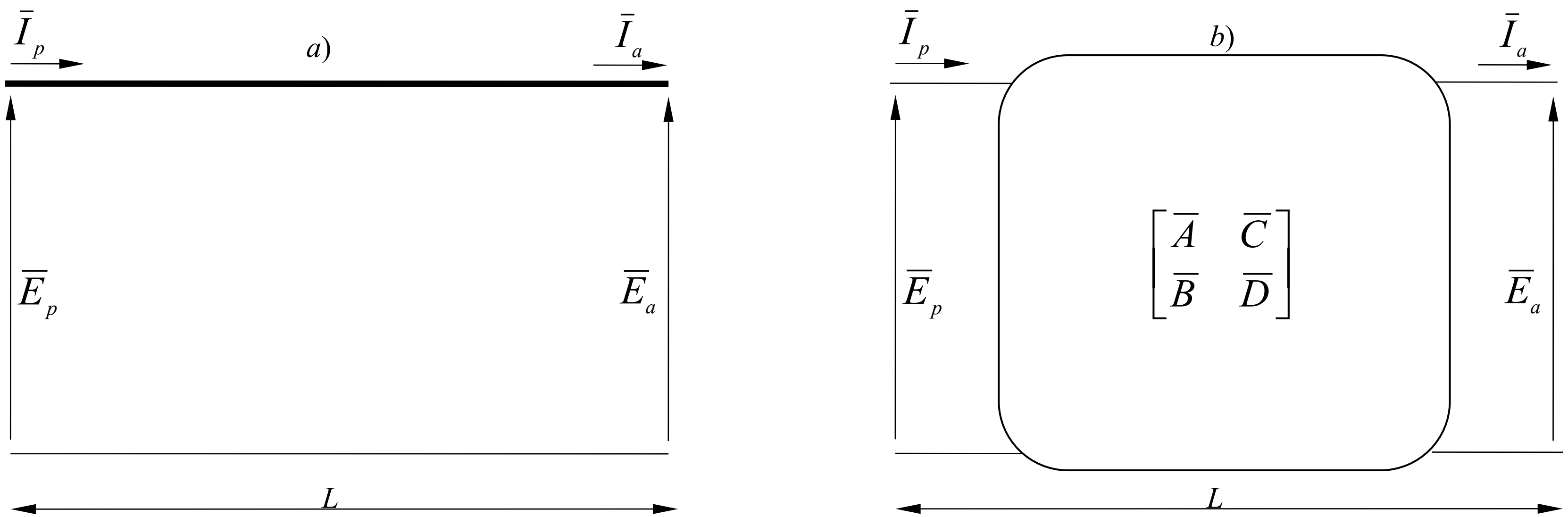
Outline

The line two-port circuit

In power systems studies, **we may not be interested in knowing the profiles of voltages and currents along the line but rather the link between voltages and currents at both ends of the line**. Therefore, it is interesting to verify if we can represent a phase of the line by a **passive two-port equivalent** with linear elements. The transfer matrix is this case must have the following form:

$$\overline{E}_p = \overline{A}\,\overline{E}_a + \overline{B}\,\overline{I}_a$$

$$\overline{I}_p = \overline{C}\,\overline{E}_a + \overline{D}\,\overline{I}_a$$



Single-phase of a line (a) represented by a two-port equivalent (b).

The line two-port circuit

Observation: it is important to note that a transmission line is a system with **two degrees of freedom**. Namely, it is characterized by two independent relationships of the four variables of interest (voltages and currents) that allow us to assess the electrical state of the network. If we consider our system with two inputs, $(\overline{E}_p, \overline{I}_p)$ and two outputs, $(\overline{E}_a, \overline{I}_a)$ variables it is possible to obtain the parameters of the matrix coefficients $\overline{A}, \overline{B}, \overline{C}, \overline{D}$.

The expressions for the four constants $\overline{A}, \overline{B}, \overline{C}, \overline{D}$ can be obtained by using the solution of the transmission line equations written for the orientation x^{-} and imposing $x = L$

$$\overline{E}_x = \overline{E}_a \cosh \overline{\gamma} x + \overline{Z}_0 \overline{I}_a \sinh \overline{\gamma} x$$

$$\overline{I}_x = \overline{I}_a \cosh \overline{\gamma} x + \frac{\overline{E}_a}{\overline{Z}_0} \sinh \overline{\gamma} x$$

$x = L$

$$\overline{E}_p = \overline{A}\,\overline{E}_a + \overline{B}\,\overline{I}_a$$

$$\overline{I}_p = \overline{C}\,\overline{E}_a + \overline{D}\,\overline{I}_a$$

$$\begin{cases} \overline{A} = \overline{D} = \cosh \overline{\gamma} L \\ \overline{B} = \overline{Z}_0 \sinh \overline{\gamma} L \\ \overline{C} = \dfrac{1}{\overline{Z}_0} \sinh \overline{\gamma} L \end{cases}$$

The four constants that are dependent on $\overline{\gamma}$ and $\overline{Z}_0$ are called **auxiliary constants** of the line.

The line two-port circuit

Mathematically, we have that $\bar{A} = \bar{D}$ and that $\bar{A}\bar{D} - \bar{B}\bar{C} = 1$.
This last is the **principle of reciprocity** of two-port equivalent circuits.
We recall that it **must hold if the two-port circuit is composed by passive constant elements, and it does not contain amplifiers**.

It is therefore necessary, and sufficient, to know only two of the four auxiliary constants to determine the two-port equivalent.

It should be emphasized that the two-port model **can not determine the shape of the voltage and current across the line but only the (linear) link between voltages and currents at the beginning and at the end of the line.**

The line two-port circuit

The auxiliary constants can be **interpreted** as follows.

- For the line in 'open-circuit' $\left(\overline{I}_a = 0\right)$ we have:

$$\overline{E}_p = \overline{A}\,\overline{E}_a \quad , \quad \overline{I}_p = \overline{C}\,\overline{E}_a$$

- For the line in 'short-circuit' $\left(\overline{E}_a = 0\right)$ we have:

$$\overline{E}_p = \overline{B}\,\overline{I}_a \quad , \quad \overline{I}_p = \overline{A}\,\overline{I}_a$$

Then, we can provide the following **interpretations of the auxiliary constants**:

- Constant $\bar{A}$ is defined as the **ratio of the voltage in the input and the one in the output of the line in 'open-circuit'** (or, as the ratio of the current in the input and the one in the output of the line in 'short-circuit'). This constant is a dimensionless number and the following nomenclature will be used:

$$a_1 + j a_2 \;\; \left(a_1 < 1 \; ; \; a_2 << 1\right) \quad \text{or} \quad A e^{j\,\alpha_A}$$

The line two-port circuit

- Constant $\bar{B}$ is defined as the **ratio between the voltage at the line start divided by the current at the line end when it is in 'short-circuit'**. It is, therefore, an impedance.

$$b_1 + j b_2 \quad \left(b_1 \approx r \cdot L = R \;\; ; \;\; b_2 \approx x \cdot L = X\right) \qquad \text{or} \qquad B e^{j\beta_B}$$

Observation: as we will see later, this constant is quite **close to the total longitudinal impedance of the line:** $\bar{B} \approx \bar{z}L = \bar{Z}$.

- Constant $\bar{C}$ is defined as the **ratio between the current at the line start divided by the voltage at the line end when it is in 'open-circuit'**. It is, therefore, an admittance.

$$c_1 + j c_2 \qquad \text{or} \qquad C e^{j\gamma_C}$$

Observation: as we will see later, this constant is quite **close to the total shunt admittance of the line**: $\bar{C} \approx \bar{y}L = \bar{Y}$. Furthermore, for actual lines, c_1 is close to zero.

The line two-port circuit

It is possible to infer the four auxiliary constants through the hyperbolic functions used for their definition. We firstly define the global parameters of the line (with L the overall length of the line).

Total longitudinal line impedance:

$$\overline{Z} = \overline{z}\, L$$

Total transversal line admittance:

$$\overline{Y} = \overline{y} L$$

Then, we have:

$$\overline{\gamma} L = \left(\sqrt{\overline{z}\overline{y}}\right) L = \overline{Z}^{\frac{1}{2}} \cdot \overline{Y}^{\frac{1}{2}}$$

$$\overline{Z}_0 = \left(\sqrt{\frac{\overline{z}}{\overline{y}}}\right) = \left(\sqrt{\frac{\overline{z}L}{\overline{y}L}}\right) = \overline{Z}^{\frac{1}{2}} \cdot \overline{Y}^{-\frac{1}{2}}$$

The line two-port circuit

It is interesting to write the **Taylor expansion** of the previous equations to determine **simplified line models** that are **function of the line type**.

$$\begin{cases} \bar{A} = \cosh \bar{\gamma} L = 1 + \dfrac{\bar{Z}\bar{Y}}{2!} + \dfrac{\bar{Z}^2\bar{Y}^2}{4!} + \ldots\ldots \\ \bar{B} = \bar{Z}_0 \sinh \bar{\gamma} L = \bar{Z}\left(1 + \dfrac{\bar{Z}\bar{Y}}{3!} + \dfrac{\bar{Z}^2\bar{Y}^2}{5!} + \ldots\ldots\right) \\ \bar{C} = \dfrac{1}{\bar{Z}_0} \sinh \bar{\gamma} L = \bar{Y}\left(1 + \dfrac{\bar{Z}\bar{Y}}{3!} + \dfrac{\bar{Z}^2\bar{Y}^2}{5!} + \ldots\ldots\right) \end{cases}$$

The **terms of the series expansion to be kept depend on the desired approximation of the line parameters in relation with the importance of the line** and, above all, as a function of the **importance of terms with the factor** $\bar{Z}^n\bar{Y}^n$. To give an idea, the factor $\bar{Z}\bar{Y} = \bar{z}\bar{y}L^2$ depends on the **square of the line length** and, for overhead lines operating at $50\,Hz$, $|\bar{z}\bar{y}| \approx 10^{-12}\left[\frac{1}{m^2}\right]$. So, the terms to be kept depend on: (i) the **line length**, (ii) the **line rated voltage** (since it **weights the importance of the shunt admittance** $\bar{Y}$) and (iii) the **line type overhead vs cable**, since it has an **influence on the magnitude of** $\bar{Y}$ (**cables have larger** $\bar{Y}$).

The line two-port circuit

It is also worth observing that the modules of the total impedances and admittances $\bar{Z}$ and $\bar{Y}$ of the line **depend on the line length beyond the specific value of** L. Indeed, **when a line is long this means that the voltage is high**. Furthermore, if a line is composed by a coaxial cable, its length is generally shorter than the corresponding overhead line with the same nominal voltage (this is because the coaxial cables produce a large amount of reactive power). In summary, the terms to be kept in the Taylor expansions must account for: the **line length**, the **line nominal voltage** and the **line type overhead or cable**.

In what follows typical Taylor series expansions are given for actual lines.

The line two-port circuit

- **Short lines**:

$L \leq 50$ km	$V \leq 30$ kV	for overhead lines;
$L \leq 03$ km	$V \leq 20$ kV	for cables.

In this case we can neglect all the transversal parameters and so:

$$\overline{A} = 1 \quad , \quad \overline{B} = \overline{Z} \quad , \quad \overline{C} = 0$$

Note that shunt capacitances are larger in coaxial cables than the corresponding overhead lines. For this reason, in short overhead lines, we may neglect the shunt parameters for line lengths larger than the corresponding cables ones.

The line two-port circuit

- **Medium-length lines**

  $50 \leq L \leq 100$ km $30 \leq V \leq 66$ kV for overhead lines;

  $03 \leq L \leq 10$ km $20 \leq V \leq 60$ kV for cables.

The Taylor expansion stops in correspondence of the first term:

$$\overline{A} = 1 \quad , \quad \overline{B} = \overline{Z} \quad , \quad \overline{C} = \overline{Y}$$

- **Long lines**

  $100 \leq L \leq 300$ km $66 \leq V \leq 132$ kV for overhead lines;

  $10 \leq L \leq 30$ km $60 \leq V \leq 66$ kV for cables.

The expansion stops in correspondence of the second term:

$$\overline{A} = 1 + \frac{\overline{Z}\,\overline{Y}}{2}, \overline{B} = \overline{Z}\left(1 + \frac{\overline{Z}\,\overline{Y}}{6}\right), \overline{C} = \overline{Y}\left(1 + \frac{\overline{Z}\,\overline{Y}}{6}\right)$$

The line two-port circuit

- For lines with nominal voltage between 132 kV and 380 kV we can use all the terms of the Taylir expansion shown on slide 28.

- For lines with nominal voltage ≥ 380 kV, we have to use the hyperbolic expressions with no approximations.

The typical parameters of a transmission line are reported on slide 33.

Important observation: for load flow calculation, it is worth reminding that any electrical element described by a two-port equivalent passive and reciprocal, can be replaced by equivalent T or Π circuits (see slide 34).

The line two-port circuit

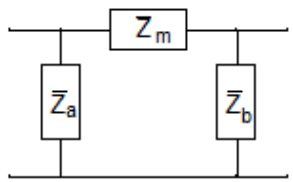
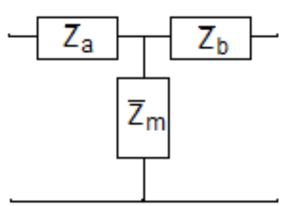
Typical transmission line parameters for medium and high voltage overhead and cable lines.

Quantity		Overhead 20 KV (1)	Overhead 380 KV (2)	Cable 20 KV (3)	Cable 380 KV (4)
Resistance **r** $[\Omega \cdot m^{-1}]$		$0,268 \cdot 10^{-3}$ (5)	$0,020 \cdot 10^{-3}$ (5)	$0,0995 \cdot 10^{-3}$ (6)	$0,025 \cdot 10^{-3}$ (6)
Conductance **g** $[S \cdot m^{-1}]$		—	$0,007 \cdot 10^{-9}$	$0,285 \cdot 10^{-9}$	$0,194 \cdot 10^{-9}$
Inductance **l** $[H \cdot m^{-1}]$		$1,1 \cdot 10^{-6}$	$0,853 \cdot 10^{-6}$	$0,309 \cdot 10^{-6}$	$0,710 \cdot 10^{-6}$
Reactance **x** $[\Omega \cdot m^{-1}]$		$0,345 \cdot 10^{-3}$	$0,268 \cdot 10^{-3}$	$0,0971 \cdot 10^{-3}$	$0,223 \cdot 10^{-3}$
Capacitance **c** $[F \cdot m^{-1}]$		$10,7 \cdot 10^{-12}$	$13,7 \cdot 10^{-12}$	$302 \cdot 10^{-12}$	$266 \cdot 10^{-12}$
Susceptance **b** $[S \cdot m^{-1}]$		$3,36 \cdot 10^{-9}$	$4,30 \cdot 10^{-9}$	$94,9 \cdot 10^{-9}$	$83,6 \cdot 10^{-9}$
$\bar{\gamma}$	Att. Const. $[m^{-1}]$	$0,393 \cdot 10^{-6}$	$0,041 \cdot 10^{-6}$	$1,42 \cdot 10^{-6}$	$0,247 \cdot 10^{-6}$
	Phase Const. $[° \cdot m^{-1}]$	$0,0659 \cdot 10^{-3}$	$0,0615 \cdot 10^{-3}$	$0,192 \cdot 10^{-3}$	$0,248 \cdot 10^{-3}$
$\bar{Z}_o$	Module $[\Omega^{-1}]$	361	250	38	52
	Phase [°]	-18.92	-2.09	-22.65	-3.13

(1) Line with Cu conductors 70 mm^2
(2) Line with AlAcr diameter 31,5 mm
(3) Tri-polar Cu cable conduttors 240 mm^2, insulation with EPR
(4) Tri-polar Cu cable 1200 mm^2, insulation with fluid oil
(5) Temperature of 20°C (6)Temperature of 85°C

The line two-port circuit

Circuit representation of transmission lines using T or Π equivalent circuits.

TYPE OF TWO-PORT EQUIVALENT	Passive two-port eq. Symetrical reciprocal	
	3	4
	$\bar{A}=\bar{D}=1+\frac{\bar{Z}_m}{\bar{Z}_a}$ $=\cosh \bar{\gamma}L$ $\bar{B}=\bar{Z}_m=\bar{Z}_0 \sinh \bar{\gamma}L$ $\bar{C}=\frac{2}{\bar{Z}_a}+\frac{\bar{Z}_m}{\bar{Z}_a^2}=\frac{1}{\bar{Z}_0}\sinh \bar{\gamma}L$	$\bar{Z}_a=\bar{Z}_b=\frac{\bar{A}+1}{\bar{C}}=$ $=\frac{2}{\bar{Y}}\frac{\bar{\gamma}L/2}{\tanh(\bar{\gamma}L/2)}$ $\bar{Z}_m=\bar{B}=\bar{Z}\frac{\sinh\bar{\gamma}L}{\bar{\gamma}L}$
	$\bar{A}=\bar{D}=1+\frac{\bar{Z}_a}{\bar{Z}_m}$ $\bar{B}=2\bar{Z}_a+\frac{\bar{Z}_a^2}{\bar{Z}_m}$ $\bar{C}=\frac{1}{\bar{Z}_m}$	$\bar{Z}_a=\bar{Z}_b=\frac{\bar{B}}{\bar{A}+1}=$ $=\frac{\bar{Z}}{2}\frac{\sinh\bar{\gamma}L}{\bar{\gamma}L}$ $\bar{Z}_m=\frac{1}{\bar{C}}=\frac{1}{\bar{Y}}\frac{\bar{\gamma}L/2}{\tanh(\bar{\gamma}L/2)}$

Outline

The transmission line equations

The line two-port circuit

Active-reactive power flow expressions in transmission lines

Characteristic and natural power

The Baum-Perrine diagram

Ferranti effect

Final remarks

Active-reactive power flow expressions in transmission lines

The expression of the apparent power at the end of the line reads:

$$\bar{S}_a = P_a + jQ_a = 3\overline{E_a}\,\underline{I}_a$$

and since

$$\bar{E}_a$$

$$\bar{E}_p = \bar{A}\bar{E}_a + \bar{B}\bar{I}_a \Rightarrow \bar{I}_a = \frac{\bar{E}_p - \bar{A}\bar{E}_a}{\bar{B}} \Rightarrow$$

$$\Rightarrow \bar{S}_a = 3\overline{E_a}\left(\frac{\underline{E}_p - \underline{A}\underline{E}_a}{\underline{B}}\right) \qquad \vartheta = \arg\left(\bar{E}_p - \bar{E}_a\right)$$

we can assume that $\bar{E}_a$ is placed on the real axis (the modeler has the freedom to choose a reference phasor). Furthermore, we define θ as the electrical angle between $\bar{E}_a$ and $\bar{E}_p$: $\theta = arg\left(\bar{E}_p - \bar{E}_a\right)$. Therefore:

$$\bar{S}_a = P_a + jQ_a = 3\frac{\overline{E_a E_p}}{\underline{B}} - 3\frac{\overline{E_a E_a A}}{\underline{B}} = 3\frac{E_a E_p}{B}\frac{e^{-j\theta}}{e^{-j\beta_B}} - 3\frac{E_a^2 A}{B}\frac{e^{-j\alpha_A}}{e^{-j\beta_B}} =$$

$$= 3\frac{E_a E_p}{B}e^{j(\beta_B - \theta)} - 3E_a^2\frac{A}{B}e^{j(\beta_B - \alpha_A)} =$$

$$\frac{3E_a E_p}{B}\cos(\beta_B - \theta) - 3E_a^2\frac{A}{B}\cos(\beta_B - \alpha_A) + j\left[\frac{3E_a E_p}{B}\sin(\beta_B - \theta) - 3E_a^2\frac{A}{B}\sin(\beta_B - \alpha_A)\right]$$

Active-reactive power flow expressions in transmission lines

From the previous expression of the apparent power, we can extrapolate the real and imaginary parts corresponding to the active and reactive power flows at the line arrival:

$$P_a = \frac{3E_aE_p}{B}\cos(\beta_B - \theta) - 3E_a^2\frac{A}{B}\cos(\beta_B - \alpha_A)$$
$$Q_a = \frac{3E_aE_p}{B}\sin(\beta_B - \theta) - 3E_a^2\frac{A}{B}\sin(\beta_B - \alpha_A)$$

By following a similar procedure, we can derive the active and reactive power flows at the line departure:

$$P_p = -\frac{3E_aE_p}{B}\cos(\beta_B + \theta) + 3E_p^2\frac{A}{B}\cos(\beta_B - \alpha_A)$$
$$Q_p = -\frac{3E_aE_p}{B}\sin(\beta_B + \theta) + 3E_p^2\frac{A}{\mathrm{B}}\sin(\beta_B - \alpha_A)$$

Active-reactive power flow expressions in transmission lines

If the line is **lossless** and **without shunt admittance**, the equations that we wrote become even simpler ($\alpha_A = 0, \left|\overline{A}\right| = 1, \beta_B = \dfrac{\pi}{2}, \overline{B} = jX$):

$$P_p = P_a = \frac{3E_p E_a}{X}\sin\theta$$

$$Q_a = \frac{3E_a}{X}\left(E_p\cos\theta - E_a\right)$$

$$Q_p = \frac{3E_p}{X}\left(E_p - E_a\cos\theta\right)$$

Outline

Characteristic and natural power

Let us suppose to **supply a load via a line and the load has an impedance equal to the line's characteristic impedance $\bar{Z}_0$.** In these conditions, the power supplied to the load is called **characteristic power** and can be computed as follows:

$$\bar{S}_{0,a} = P_{0,a} + jQ_{0,a} = 3\bar{E}_a \underline{I}_a = 3\bar{E}_a \frac{\underline{E}_a}{\underline{Z}_a} =$$

$$= \frac{V_a^2}{Z_0\left(\cos\psi - j\sin\psi\right)} = \frac{V_a^2}{Z_0}\left(\cos\psi + j\sin\psi\right)$$

Observation: the argument Ψ of Z_0 is negative (see the table on slide 33). Therefore, the characteristic power has a reactive power that is negative.

Characteristic and natural power

Let us suppose that the line is lossless: $r = g = 0$ →

$$\bar{Z}_0 = \sqrt{\frac{r + j\omega l'}{g + j\omega c'}} = \sqrt{\frac{l'}{c'}}$$

In these ideal conditions, the characteristic impedance is a real number called **natural or surge impedance.**
Also, the delivered power to the load is only real (i.e., active power) and it is called **natural power**:

$$\bar{S}_{0,a} = P_{0,a} = \frac{V_a^2}{Z_0}$$

Characteristic and natural power

In these conditions we have that the voltage and current profiles obtained for the orientation x^{-}, have an easier expression:

$$\bar{E}_x = \frac{1}{2}\left(\bar{E}_a + \bar{Z}_0\bar{I}_a\right)e^{\bar{\gamma}x} + \cancel{\frac{1}{2}\left(\bar{E}_a - \bar{Z}_0\bar{I}_a\right)e^{-\bar{\gamma}x}} = \frac{1}{2}\left(\bar{E}_a + \bar{Z}_0\bar{I}_a\right)e^{\bar{\gamma}x} = \bar{E}_a e^{\bar{\gamma}x}$$

$$\bar{I}_x = \frac{1}{2}\left(\bar{I}_a + \frac{\bar{E}_a}{\bar{Z}_0}\right)e^{\bar{\gamma}x} + \cancel{\frac{1}{2}\left(\bar{I}_a - \frac{\bar{E}_a}{\bar{Z}_0}\right)e^{-\bar{\gamma}x}} = \frac{1}{2}\left(\bar{I}_a + \frac{\bar{E}_a}{\bar{Z}_0}\right)e^{\bar{\gamma}x} = \bar{I}_a e^{\bar{\gamma}x}$$

Furthermore, we have that the following equality holds $\forall x$:

$$\frac{\bar{E}_x}{\bar{I}_x} = \frac{\bar{E}_a e^{\bar{\gamma}x}}{\bar{I}_a e^{\bar{\gamma}x}} = \bar{Z}_0$$

Characteristic and natural power

For $x = L$, and for the orientation x^-, we have that $\bar{E}_x|_{x=L} = \bar{E}_p$ and $\bar{I}_x|_{x=L} = \bar{I}_p$, therefore, since $\frac{\bar{E}_x}{\bar{I}_x} = \bar{Z}_0$ also for $x = L$, we get the same result of slide 20:

$$\bar{E}_x = \cancel{\frac{1}{2}\left(\bar{E}_p - \bar{Z}_0 \bar{I}_p\right) e^{\bar{\gamma} x}} + \frac{1}{2}\left(\bar{E}_p + \bar{Z}_0 \bar{I}_p\right) e^{-\bar{\gamma} x}$$

$$\bar{I}_x = \cancel{\frac{1}{2}\left(\bar{I}_p - \frac{\bar{E}_p}{\bar{Z}_0}\right) e^{\bar{\gamma} x}} + \frac{1}{2}\left(\bar{I}_p + \frac{\bar{E}_p}{\bar{Z}_0}\right) e^{-\bar{\gamma} x}$$

In other words, **a line supplying a load equal to its characteristic impedance has the same voltage and current profiles of the same line that is infinitely long.**
Pay attention that this statement is true for $0 \leq x \leq L$.

Characteristic and natural power

Furthermore, the condition

$$\frac{\bar{E}_x}{\bar{I}_x} = \frac{\bar{E}_a e^{\bar{\gamma} x}}{\bar{I}_a e^{\bar{\gamma} x}} = \bar{Z}_0$$

Involves that **voltage and the current at each generic position x of the line are characterised by a ratio and phase displacement that constant.** Additionally, we have that:

$$\frac{\bar{E}_x}{\bar{I}_x} = Z_0 = \sqrt{\frac{l'}{c'}} \rightarrow \omega c' dx E_x^2 = \omega l' dx I_x^2$$

Namely, for a lossless line delivering the natural power, at each position x, the reactive power generated by the line infinitesimal shunt capacitance $\omega c' dx E_x^2$ is equal to the reactive power absorbed by the line infinitesimal series inductance $\omega l' dx I_x^2$.
In other words, the line is reactive balanced.

Characteristic and natural power

Furthermore, for a **lossless line delivering the natural power** we have:

$$\bar{E}_x = \bar{E}_a e^{\bar{\gamma} x} = \bar{E}_a e^{j\beta x}$$
$$\bar{I}_x = \bar{I}_a e^{\bar{\gamma} x} = \bar{I}_a e^{j\beta x}$$

Namely, voltage and current phasors along the line are simply **rotating without any attenuation.**

For **high voltage lines**, the parameters r, g are usually negligible with respect to $\omega l'$ and $\omega c'$ (see slide 33 for 380 kV lines). **Therefore, when one of these lines operates nearby the characteristic power, it is also true that it is operating close to the natural power**. **For medium voltage lines this is observation is not true**.

Characteristic and natural power

Rated voltage V_n[kV]	Characteristic impedance Z_0 [Ohm]	Natural power P_0 [MW]
20	400	1
50	400	6
132	400	44
220	390	124
380	260	560
700	240	2000
1000	240	4000

Outline

The Baum-Perrine diagram

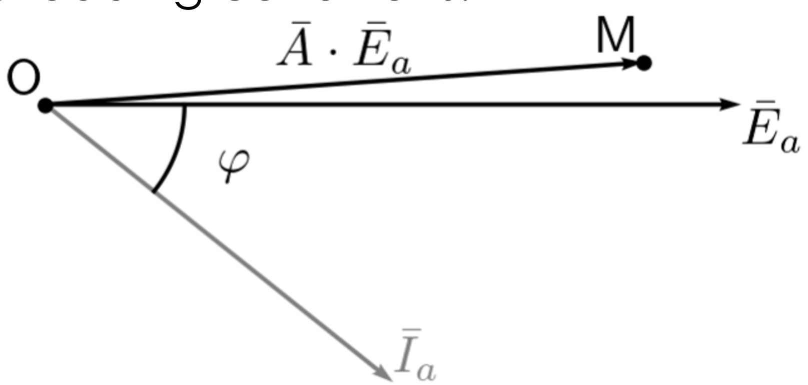
Step-by-step construction of the diagram

The Baum-Perrine diagram allows to **graphically interpret the link between the line active/reactive power flows and the voltage phasors at line terminals.**

In what follows, we start by fixing the voltage phasor $\bar{E}_a$.

Step by step Construction:

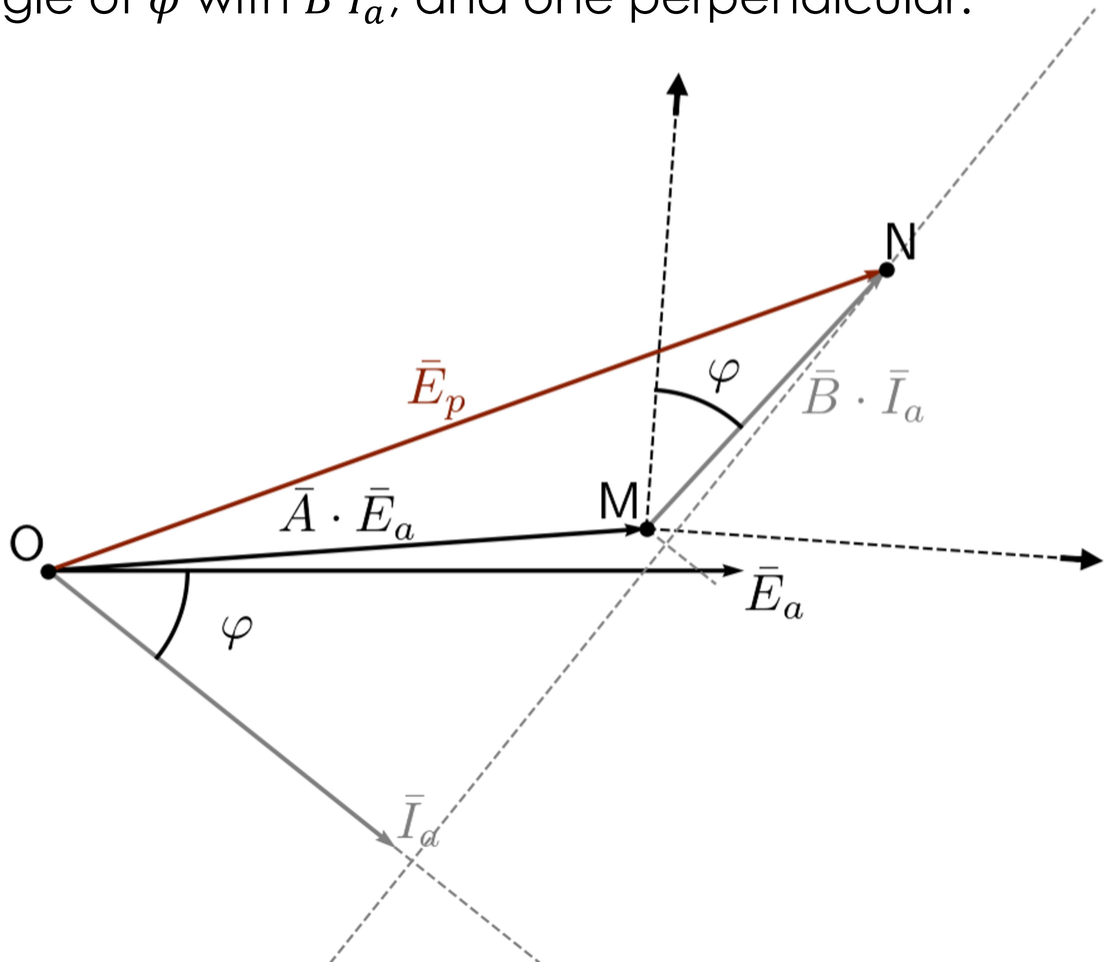
1. Draw the voltage at the end of the line $\bar{E}_a$. Since it is our reference, we place it on the real axis.
2. Draw the current phasor at the end of the line $\bar{I}_a$.
3. Draw $\bar{A}\,\bar{E}_a$, this corresponds to the voltage at the start if the line is in no-load conditions.
4. Name the end of $\bar{A}\,\bar{E}_a$ as the point M. Note that this point only depends on the physical characteristics of the line (lengths, voltage level, etc) and not on the loading conditions.



The Baum-Perrine diagram

Step-by-step construction of the diagram

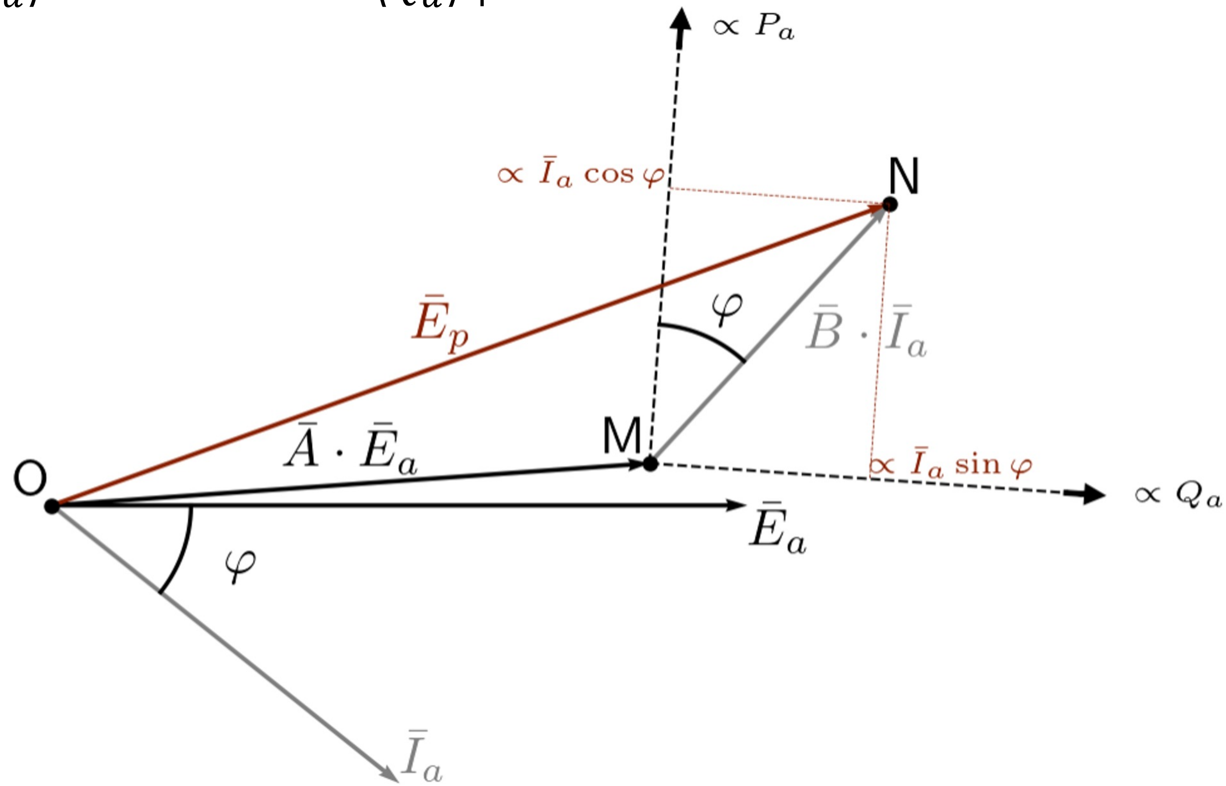
4. From M draw $\bar{B}\,\bar{I}_a$ and obtain $\bar{E}_p$ since $\bar{E}_p = \bar{A}\,\bar{E}_a + \bar{B}\,\bar{I}_a$.
5. Name the end of $\bar{E}_p$ as the point N. From N, draw two axis, one forming an angle of φ with $\bar{B}\,\bar{I}_a$, and one perpendicular.



The Baum-Perrine diagram

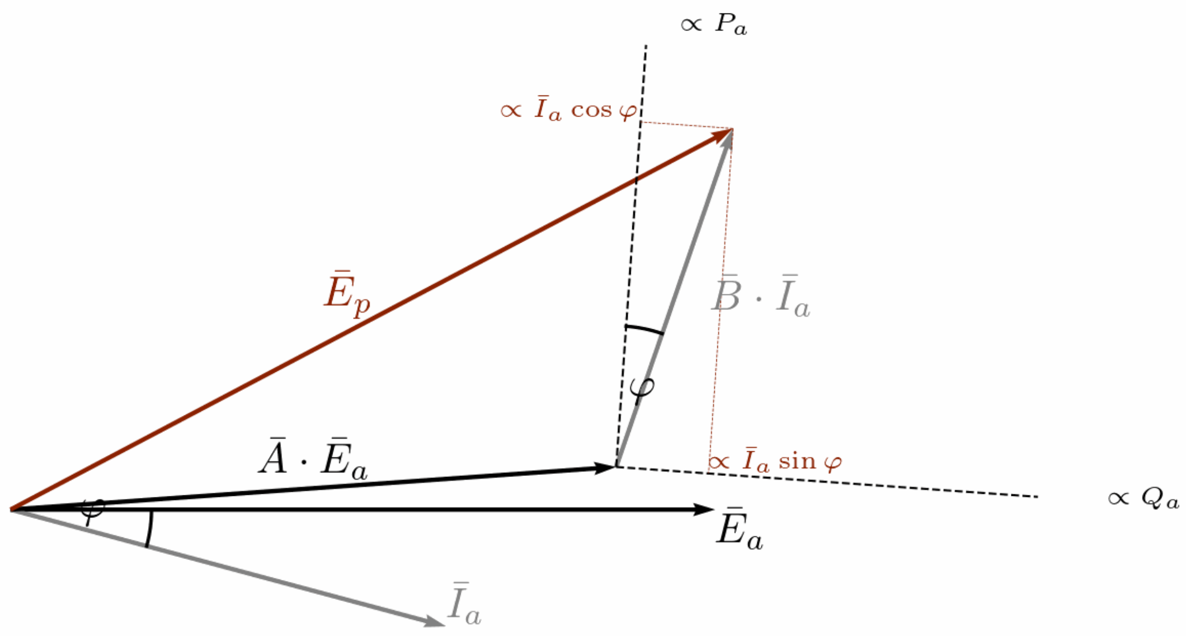
Step-by-step construction of the diagram

7. The two axes have the following characteristics:
   - They do not depend on the loading conditions of the line.
   - The projections of N onto the two axes are proportional to the active (P_a) and reactive (Q_a) powers of the line delivered at the arrival.



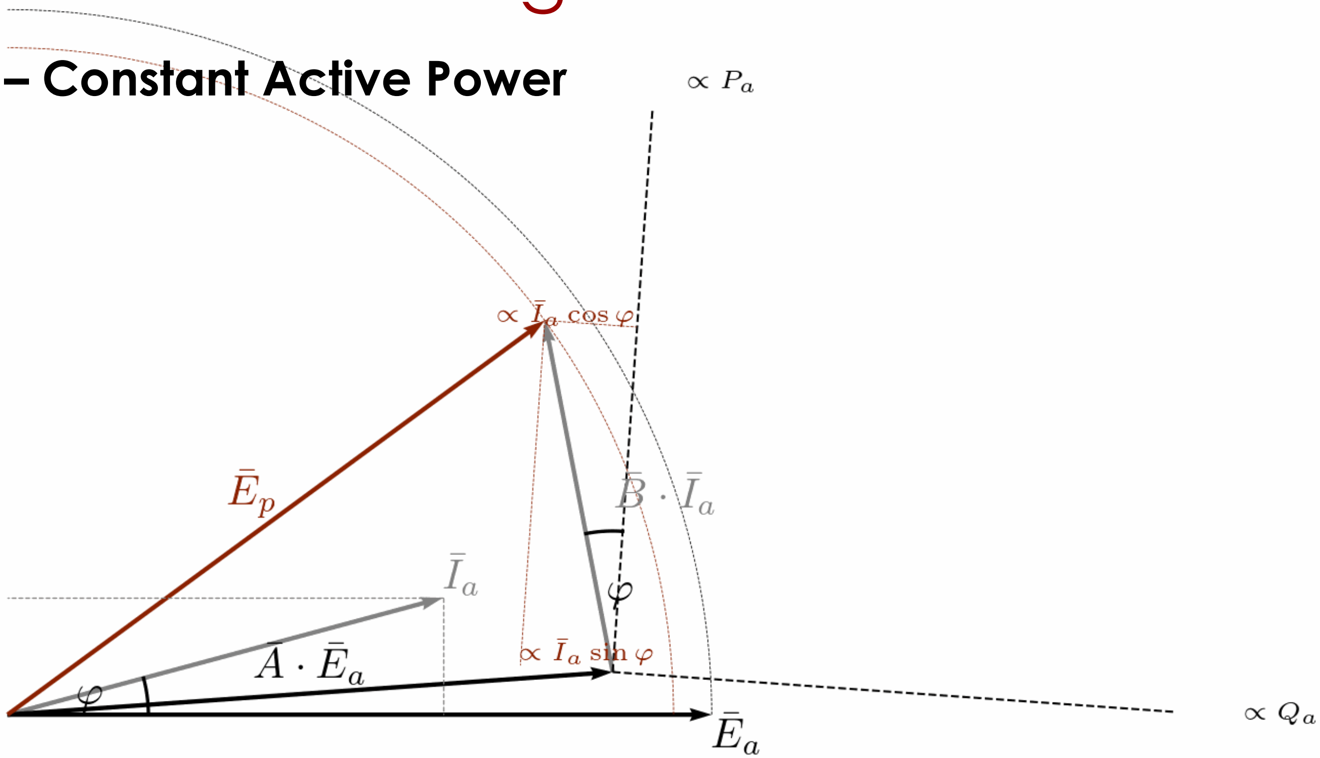
The Baum-Perrine diagram

Animation – Fixed Axes



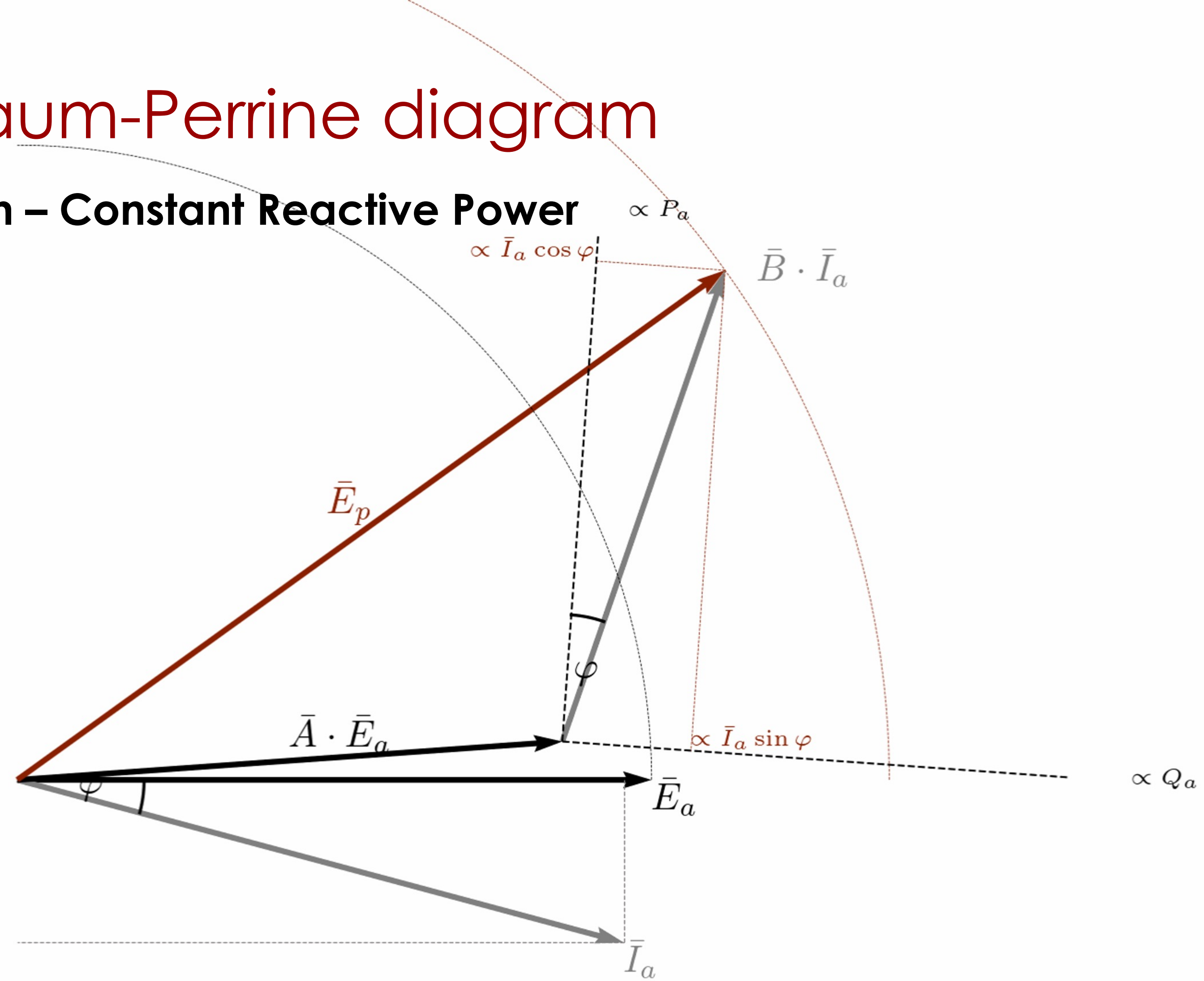
The Baum-Perrine diagram

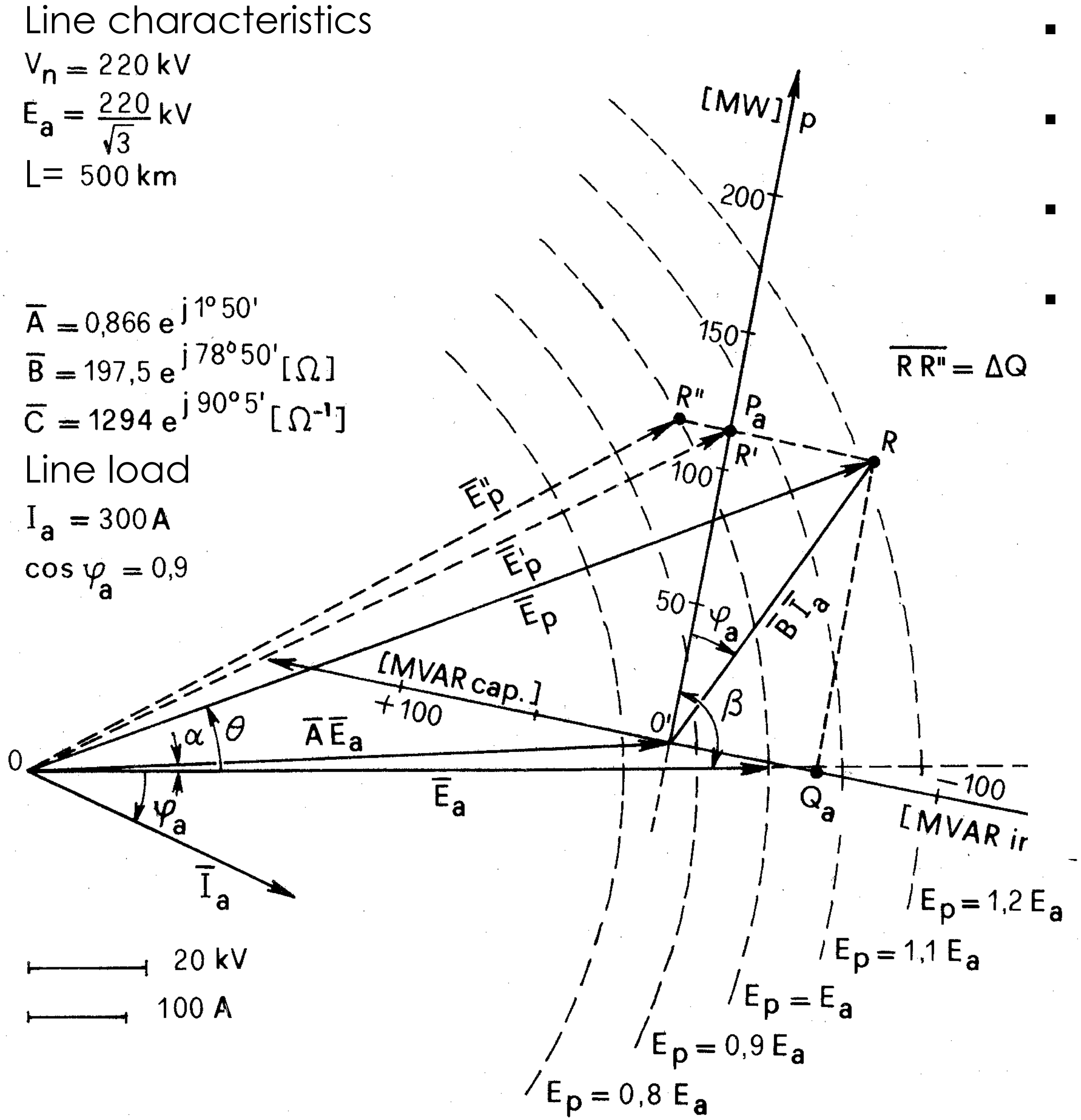
Animation – Constant Active Power



The Baum-Perrine diagram

Animation – Constant Reactive Power





- No-load: $E_p < E_a$ (Ferranti effect)
- $P_a + jQ_a$ (operation point R) → voltage drop = 21%
- P_a (operation point R') → voltage drop = 6%
- To obtain a null voltage drop (still delivering P_a) we need the phasor E_p in R". This corresponds having a capacitive load that generates the reactive power ΔQ=RR" where the fraction RR' is absorbed by the load while the fraction R'R" is injected by the line and is used to compensate the voltage drop produced by the transfer of active power to the load.

Outline

The Ferranti effect

The Ferranti effect refers to th**e phenomenon where the voltage at the end of a (long, high-voltage) transmission line is higher than the voltage at its start**, **particularly under light load or no-load conditions**.

This phenomenon is of interest to:

1. Model the behavior of the line when it is **energized after it has been kept offline due to maintenance reasons** so, the line load is zero.
2. Understand **criticalities occurring during the restoration process of a power system following a black out**. Indeed, after a black out, the grid/line loads are zero.
3. **Quantify the reactive power produced by the line when it is in no-load conditions**.

The Ferranti effect

Let us recall the solutions of the transmission line equations for the reference x^- and using hyperbolic functions:

$$\bar{E}_x = \bar{E}_a \cosh \bar{\gamma} x + \bar{Z}_0 \, \bar{I}_a \sinh \bar{\gamma} x$$

$$\bar{I}_x = \bar{I}_a \cosh \bar{\gamma} x + \frac{\bar{E}_a}{\bar{Z}_0} \sinh \bar{\gamma} x$$

Let us assume the **line to be open** at the end, i.e. $\bar{I}_a = 0$ to study the relation between the voltage at the start and the end of the line, we obtain:

$$\bar{E}_p = \bar{E}_a \cosh \bar{\gamma} L$$

$$\bar{I}_p = \frac{\bar{E}_a}{\bar{Z}_0} \sinh \bar{\gamma} L = \frac{\bar{E}_p \sinh \bar{\gamma} L}{\bar{Z}_0 \cosh \bar{\gamma} L}$$

The Ferranti effect

From the solution of the transmission line equations with reference x^+:

$$\bar{E}_x = \bar{E}_p \cosh \bar{\gamma} x - \bar{Z}_0\, \bar{I}_p \sinh \bar{\gamma} x$$

we replace the expression of $\bar{I}_p$ obtained in the previous slide:

$$\bar{E}_x = \bar{E}_p \cosh \bar{\gamma} x - \bar{Z}_0 \frac{\bar{E}_p \sinh \bar{\gamma} L}{\bar{Z}_0 \cosh \bar{\gamma} L} \sinh \bar{\gamma} x = \bar{E}_p \cosh \bar{\gamma} x - \bar{E}_p \frac{\sinh \bar{\gamma} L}{\cosh \bar{\gamma} L} \sinh \bar{\gamma} x$$

$$= \frac{\bar{E}_p}{\cosh \bar{\gamma} L} (\cosh \bar{\gamma} x \cosh \bar{\gamma} L - \sinh \bar{\gamma} L \sinh \bar{\gamma} x)$$

$$= \frac{\bar{E}_p}{\cosh \bar{\gamma} L} \cosh \bar{\gamma} (L - x)$$

We have obtained an expression that contains only the voltage along the line $\bar{E}_x$ and the voltage at the beginning of the line $\bar{E}_p$.

The Ferranti effect

$$\bar{E}_x = \frac{\bar{E}_p}{\cosh \bar{\gamma} L} \cosh \bar{\gamma}(L - x)$$

And, equivalently, for the current:

$$\bar{I}_x = \frac{\bar{E}_p}{\bar{Z}_0 \cosh \bar{\gamma} L} \sinh \bar{\gamma}(L - x)$$

If the line is also **lossless** $(r = g = 0)$, we have:

$$\bar{\gamma} = \sqrt{(r + j\omega l)(g + j\omega c)} = \pm j\omega\sqrt{lc}; \qquad \bar{Z}_0 = \sqrt{\frac{(r + j\omega l)}{(g + j\omega c)}} = \sqrt{\frac{l}{c}}$$

and if $\bar{\gamma}$ has only the imaginary component, $\cosh \mathrm{x} = \cos \mathrm{x}$, so we get:

$$\bar{E}_x = \frac{\bar{E}_p}{\cos \omega L\sqrt{lc}} \cos \omega(L - x)\sqrt{lc}$$

$$\bar{I}_x = \frac{\bar{E}_p}{\sqrt{\frac{l}{c}} \cos \omega L\sqrt{lc}} \sin \omega(L - x)\sqrt{lc}$$

The Ferranti effect

$$\bar{E}_x = \frac{\bar{E}_p}{\cos\omega L\sqrt{lc}}\cos\omega(L-x)\sqrt{lc} \qquad \bar{I}_x = \frac{\bar{E}_p}{\sqrt{\frac{l}{c}}\cos\omega L\sqrt{lc}}\sin\omega(L-x)\sqrt{lc}$$

The last two equations show that for an **unloaded line**:

- $\bar{E}_x$ is in phase with $\bar{E}_p$, and $\bar{I}_x$ is in leading quadrature with $\bar{E}_x$ at every point along the line.
- The RMS value of the voltage and current varies along the line according to a cosine and sine laws, respectively.
- There are nodes, i.e. points where the voltage is **permanently zero**) and antinodes, i.e. points where the voltage is **maximum.** A similar situation applies to the current.
- Obviously, the distance between two points where $\bar{E}_x$ (or $\bar{I}_x$) is identical is equal to the wavelength λ. Indeed, we have that the argument of the $\cos$ and $\sin$ functions has to be equal to 2π, namely: $\omega\lambda\sqrt{lc} = 2\pi$, so $\lambda = \frac{2\pi}{\omega\sqrt{lc}}$.

The Ferranti effect

- For the case under consideration, antinodes and nodes of $\bar{E}_x$ and $\bar{I}_x$ are spaced by $\lambda/2$;
- Antinodes of $\bar{E}_x$ are spaced from the nearest antinodes of $\bar{I}_x$ by $\lambda/4$;
- In high-voltage transmission lines ($L < \lambda/4$), in the case of lossless lines, we have (recall that $\lambda = \frac{2\pi}{\omega\sqrt{lc}}$):

$$\bar{E}_a = \frac{\bar{E}_p}{A} = \frac{\bar{E}_p}{\cos \omega L\sqrt{lc}} = \frac{\bar{E}_p}{\cos 2\pi \frac{L}{\lambda}}$$

Thus, for $0 \leq L \leq \lambda/4$ ($\lambda/4 = 1500$ km), it implies that $0 \leq 2\pi\frac{L}{\lambda} \leq \frac{\pi}{2} \Rightarrow \cos 2\pi\frac{L}{\lambda}$ decreases from 1 to 0, and thus the **voltage** at the (open) **end of the line is always larger than the voltage (applied) at the origin**.

Since, in practice, actual lengths of power systems lines are way shorter than 1500 km, we always have this condition when the line is unloaded.

The Ferranti effect

- If we consider the losses, for $L \to \frac{\lambda}{4}$, $\bar{E}_a$ still increases significantly, and the over-voltages at the end of the line can be much larger than those compatible with the line's insulation. This phenomenon was first observed in 1887 during the installation of underground cables in London by **Sebastian Ziani de Ferranti** on 10 kV AC power distribution systems, and thus it is called the **Ferranti effect**.
- The following table shows the overvoltage factor $\frac{Ea}{Ep} = \frac{1}{A}$ for various lengths of a typical high voltage line at 50 Hz when unloaded.

L (km)	A	$E_a/E_p = 1/A$
100	0.995	1.005
200	0.976	1.025
300	0.95	1.053
400	0.913	1.095
500	0.866	1.155
600	0.808	1.238
800	0.668	1.497
1000	0.498	2.008

In practice, to avoid such issues, overhead lines at 50 Hz should not exceed a length of 600-700 km (500-600 km at 60 Hz).

Outline

The transmission line equations

The line two-port circuit

Active-reactive power flow expressions in transmission lines

Characteristic and natural power

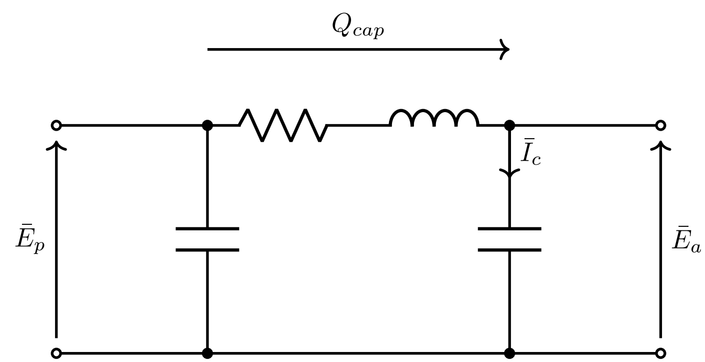
The Baum-Perrine diagram

The Ferranti effect

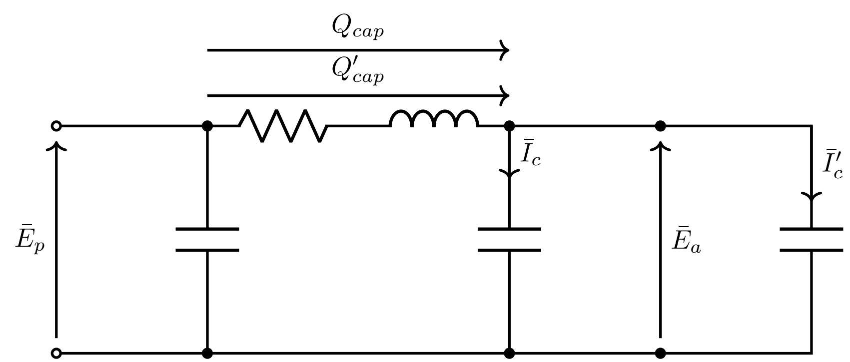
Final remarks

Final Remarks

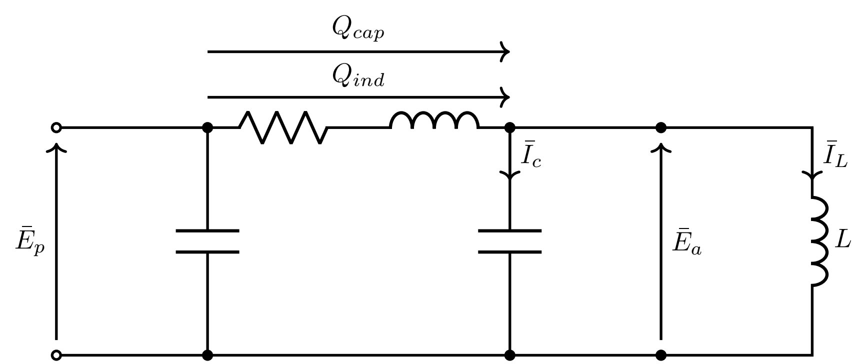
1. **Line with no load:** $|\bar{E}_a| > |\bar{E}_p|$ due to the Ferranti effect. The line behaves essentially as a capacitor since the value of shunt capacitive impedances is way bigger than the series line impedance.



2. **Lien with purely capacitive load:** the effect of the previous case is amplified. As a result,$|\bar{E}_a| \gg |\bar{E}_p|$. This condition should be avoided.

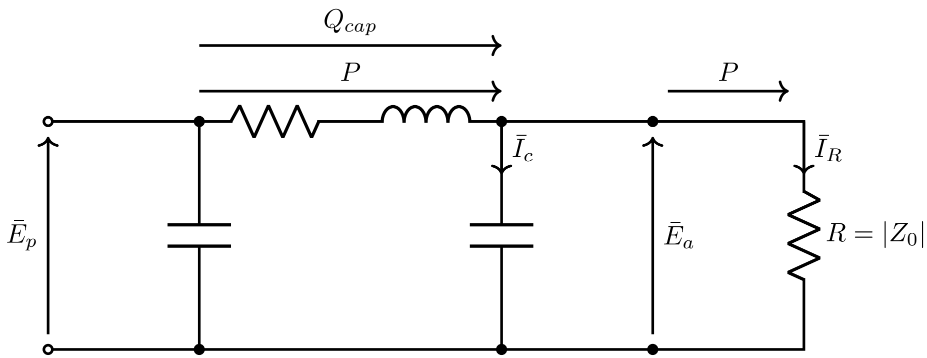


3. **Line with purely inductive load:**
   If $I_L = I_c$, then $|\bar{E}_a| \approx |\bar{E}_p|$
   If $I_L > I_c$, then $|\bar{E}_a| < |\bar{E}_p|$
   If $I_L < I_c$, then $|\bar{E}_a| > |\bar{E}_p|$

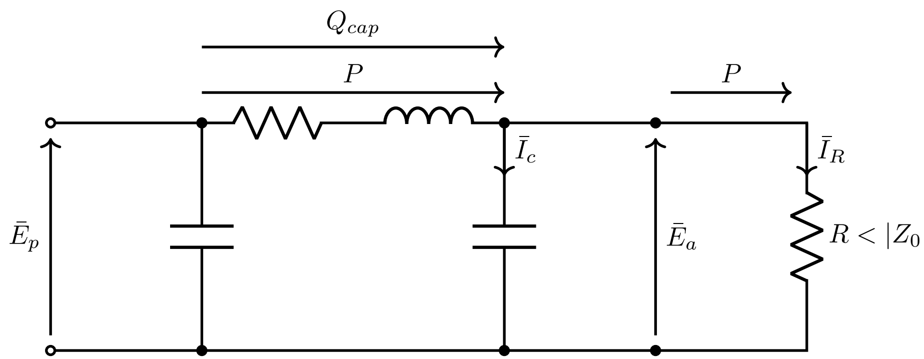


Final Remarks

4. **Line with a purely resistive load equal to the characteristic impedance $R = Z_0$ (assumed real for losses line):** the voltage drop caused by the flow of active power is almost compensated by the voltage rise due to the flow of capacitive power. As a result, $|\bar{E}_a|$ is slightly less than $|\bar{E}_p|$.

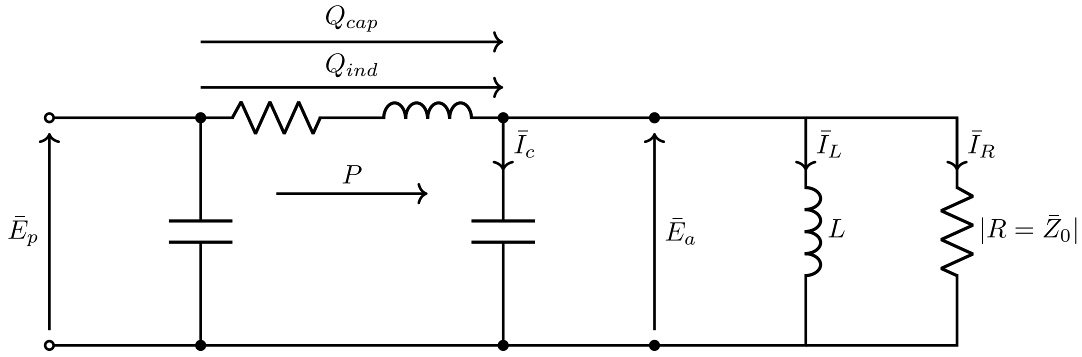


5. **Line with a purely resistive load lower than the characteristic impedances (assumed real for losses line):** the active power flow is larger than the characteristic one. The voltage drop caused by the flow of active power is larger than the voltage rise due to the flow of capacitive power. As a result, $|\bar{E}_a| < |\bar{E}_p|$.

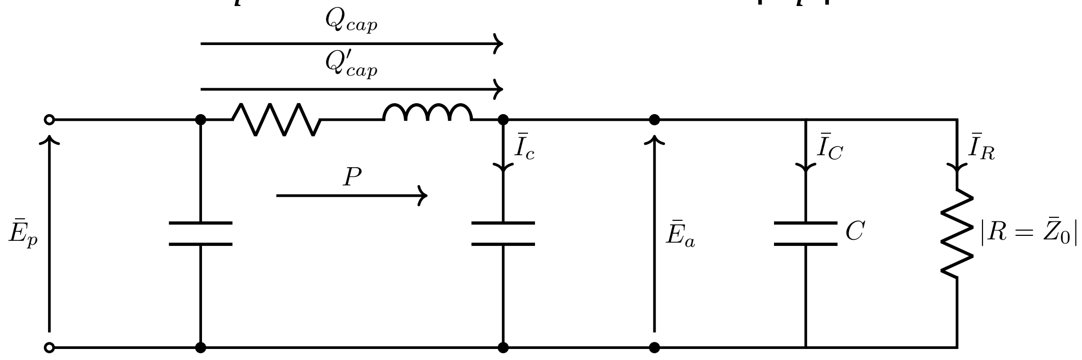


Final Remarks

6. **Line with a resistive/inductive load where $R = Z_0$:** the voltage drop due to the flow of active power P is approximately compensated by the voltage rise due to the flow of capacitive power Q_{cap}. The remaining voltage drop is due to the flow of inductive power Q_{ind} and thus $|\bar{E}_a| < |\bar{E}_p|$.



7. **Line with a resistive/capacitive load where $R = Z_0$:** as in the previous case, the effects of P and Q_{cap} are partially compensated. The remaining voltage rise is due to the flow of Q'_{cap}, and therefore$|\bar{E}_a| > |\bar{E}_p|$.



Final Remarks

In transmission lines where the **longitudinal inductance is larger compared to other parameters**, the following power transfer mechanisms are predominant (i.e., equations in slide 39):

- the **active power flow** through an electrical line is linked directly to the **angle** θ (phase difference between the voltage phasors in the end and in the beginning of the line);
- the **reactive power flow,** and its sign, is strictly linked to the **difference of the voltage magnitudes** at the two line termnals.

In transmission lines where the **parameters have equal weight** (for instance in **distribution networks where the ratio** r/x_l**'>1**) the mechanisms that govern active and reactive power flows described by the above two bullet points are **no longer valid and a cross-link between amplitude and angles of the voltage phasors at the line terminations and active/reactive power exists meaning that equations on slide 38 cannot be approximated**.